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Name of the Student: **Branch:**

Unit – I (Random Variables)

1) **Discrete random variable:**
A random variable whose set of possible values is either finite or countably infinite is called discrete random variable.
Eg: (i) Let X represent the sum of the numbers on the 2 dice, when two dice are thrown. In this case the random variable X takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. So X is a discrete random variable.
(ii) Number of transmitted bits received in error.
2) **Continuous random variable:**
A random variable X is said to be continuous if it takes all possible values between certain limits.
Eg: The length of time during which a vacuum tube installed in a circuit functions is a continuous random variable, number of scratches on a surface, proportion of defective parts among 1000 tested, number of transmitted in error.
3)

Sl.No.	Discrete random variable	Continuous random variable
1	$\sum_{i=-\infty}^{\infty} p(x_i) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$
2	$F(x) = P[X \le x]$	$F(x) = P[X \le x] = \int_{-\infty}^{x} f(x)dx$
3	$\text{Mean} = E[X] = \sum_i x_i p(x_i)$	$\text{Mean} = E[X] = \int_{-\infty}^{\infty} xf(x)dx$
4	$E[X^2] = \sum_i x_i^2 p(x_i)$	$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx$

5	$\text{Var}(X)=E(X^2)-[E(X)]^2$	$\text{Var}(X)=E(X^2)-[E(X)]^2$
6	Moment = $E[X^r]=\sum_i x_i^r p_i$	Moment = $E[X^r]=\int_{-\infty}^{\infty} x^r f(x)dx$
7	M.G.F $M_X(t)=E[e^{tX}]=\sum_x e^{tx}p(x)$	M.G.F $M_X(t)=E[e^{tX}]=\int_{-\infty}^{\infty} e^{tx} f(x)dx$

4) $E(aX+b)=aE(X)+b$
5) $\text{Var}(aX+b)=a^2\text{Var}(X)$
6) $\text{Var}(aX\pm bY)=a^2\text{Var}(X)+b^2 Var(Y)$
7) $\textbf{Standard Deviation}=\sqrt{\text{Var}(X)}$
8) $f(x)=F'(x)$
9) $p(X>a)=1-p(X\le a)$
10) $p(A/B)=\dfrac{p(A\cap B)}{p(B)},\quad p(B)\neq 0$
11) If A and B are independent, then $p(A\cap B)=p(A)\cdot p(B)$.
12) 1st Moment about origin = $E[X]=\left[M_X'(t)\right]_{t=0}$ (Mean)

2nd Moment about origin = $E[X^2]=\left[M_X''(t)\right]_{t=0}$

The co-efficient of $\dfrac{t^r}{r!}=E[X^r]$ (rth Moment about the origin)

13) **Limitation of M.G.F:**
 i) A random variable X may have no moments although its m.g.f exists.
 ii) A random variable X can have its m.g.f and some or all moments, yet the m.g.f does not generate the moments.
 iii) A random variable X can have all or some moments, but m.g.f does not exist except perhaps at one point.

14) **Properties of M.G.F:**
 i) If Y = aX + b, then $M_Y(t)=e^{bt}M_X(at)$.
 ii) $M_{cX}(t)=M_X(ct)$, where c is constant.
 iii) If X and Y are two independent random variables then $M_{X+Y}(t)=M_X(t)\cdot M_Y(t)$.

15) P.D.F, M.G.F, Mean and Variance of all the distributions:

Sl. No.	Distribution	P.D.F ($P(X=x)$)	M.G.F	Mean	Variance
1	Binomial	$nc_x p^x q^{n-x}$	$(q+pe^t)^n$	np	npq

2	Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}$	$e^{\lambda(e^t-1)}$	λ	λ
3	Geometric	$q^{x-1}p$ (or) $q^x p$	$\frac{pe^t}{1-qe^t}$	$\frac{1}{p}$	$\frac{q}{p^2}$
4	Negative Binomial	$(x+k-1)C_{k-1}p^k p^x$	$\left(\frac{p}{1-qe^t}\right)^k$	$\frac{kq}{p}$	$\frac{kq}{p^2}$
5	Uniform	$f(x)=\begin{cases}\frac{1}{b-a}, & a<x<b\\ 0, & \text{otherwise}\end{cases}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
6	Exponential	$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x>0,\lambda>0\\ 0, & \text{otherwise}\end{cases}$	$\frac{\lambda}{\lambda-t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
7	Gamma	$f(x)=\frac{e^{-x}x^{\lambda-1}}{\Gamma(\lambda)},\ 0<x<\infty,\lambda>0$	$\frac{1}{(1-t)^\lambda}$	λ	λ
8	Weibull	$f(x)=\alpha\beta x^{\beta-1}e^{-\alpha x^\beta},\ x>0,\alpha,\beta>0$			

16) Memoryless property of exponential distribution

$$P\left(X>S+t/X>S\right)=P\left(X>t\right).$$

Unit – II (Two Dimensional Random Variables)

1) $\sum_i\sum_j p_{ij}=1$ (Discrete random variable)

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy=1$ (Continuous random variable)

2) Conditional probability function X given Y, $P\left\{X=x_i/Y=y_i\right\}=\frac{P(x,y)}{P(y)}$.

Conditional probability function Y given X, $P\left\{Y=y_i/X=x_i\right\}=\frac{P(x,y)}{P(x)}$.

$$P\left\{X<a/Y<b\right\}=\frac{P\left(X<a,Y<b\right)}{P(Y<b)}$$

3) Conditional density function of X given Y, $f(x/y) = \dfrac{f(x,y)}{f(y)}$.

Conditional density function of Y given X, $f(y/x) = \dfrac{f(x,y)}{f(x)}$.

4) If X and Y are independent random variables then

$f(x,y) = f(x).f(y)$ (for continuous random variable)

$P(X = x, Y = y) = P(X = x).P(Y = y)$ (for discrete random variable)

5) Joint probability density function $P(a \le X \le b, c \le Y \le d) = \int_c^d \int_a^b f(x,y)\,dxdy$.

$$P(X < a, Y < b) = \int_0^b \int_0^a f(x,y)\,dxdy$$

6) Marginal density function of X, $f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Marginal density function of Y, $f(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

7) $P(X + Y \ge 1) = 1 - P(X + Y < 1)$

8) **Correlation co – efficient (Discrete):** $\rho(x,y) = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$Cov(X,Y) = \dfrac{1}{n}\sum XY - \bar{X}\bar{Y}$, $\sigma_X = \sqrt{\dfrac{1}{n}\sum X^2 - \bar{X}^2}$, $\sigma_Y = \sqrt{\dfrac{1}{n}\sum Y^2 - \bar{Y}^2}$

9) **Correlation co – efficient (Continuous):** $\rho(x,y) = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$Cov(X,Y) = E(X,Y) - E(X)E(Y)$, $\sigma_X = \sqrt{Var(X)}$, $\sigma_Y = \sqrt{Var(Y)}$

10) If X and Y are uncorrelated random variables, then $Cov(X,Y) = 0$.

11) $E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$, $E(Y) = \int_{-\infty}^{\infty} yf(y)\,dy$, $E(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dxdy$.

12) Regression for Discrete random variable:

Regression line X on Y is $x - \bar{x} = b_{xy}(y - \bar{y})$, $b_{xy} = \dfrac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2}$

Regression line Y on X is $y - \bar{y} = b_{yx}\left(x - \bar{x}\right), b_{yx} = \dfrac{\sum\left(x-\bar{x}\right)\left(y-\bar{y}\right)}{\sum\left(x-\bar{x}\right)^2}$

Correlation through the regression, $\rho = \pm\sqrt{b_{XY}.b_{YX}}$ **Note:** $\rho(x,y) = r(x,y)$

13) Regression for Continuous random variable:

Regression line X on Y is $x - E(x) = b_{xy}\left(y - E(y)\right)$, $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$

Regression line Y on X is $y - E(y) = b_{yx}\left(x - E(x)\right)$, $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$

Regression curve X on Y is $x = E\left(x/y\right) = \int_{-\infty}^{\infty} x\, f\left(x/y\right)\, dx$

Regression curve Y on X is $y = E\left(y/x\right) = \int_{-\infty}^{\infty} y\, f\left(y/x\right)\, dy$

14) Transformation Random Variables:

$f_Y(y) = f_X(x)\left|\dfrac{dx}{dy}\right|$ (One dimensional random variable)

$f_{UV}(u,v) = f_{XY}(x,y)\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ (Two dimensional random variable)

Unit – III (Random Processes)

1) **Random Process:**

A random process is a collection of random variables {X(s,t)} that are functions of a real variable, namely time 't' where s ∈ S and t ∈ T.

2) **Classification of Random Processes:**

We can classify the random process according to the characteristics of time t and the random variable X. We shall consider only four cases based on t and X having values in the ranges $-\infty < t < \infty$ and $-\infty < x < \infty$.

Continuous random process

Continuous random sequence

Discrete random process

Discrete random sequence

Continuous random process:

If X and t are continuous, then we call X(t) , a Continuous Random Process.

Example: If X(t) represents the maximum temperature at a place in the interval (0,t), {X(t)} is a Continuous Random Process.

Continuous Random Sequence:

A random process for which X is continuous but time takes only discrete values is called a Continuous Random Sequence.

Example: If X_n represents the temperature at the end of the nth hour of a day, then $\{X_n, 1 \le n \le 24\}$ is a Continuous Random Sequence.

Discrete Random Process:

If X assumes only discrete values and t is continuous, then we call such random process {X(t)} as Discrete Random Process.

Example: If X(t) represents the number of telephone calls received in the interval (0,t) the {X(t)} is a discrete random process since S = {0,1,2,3, . . . }

Discrete Random Sequence:

A random process in which both the random variable and time are discrete is called Discrete Random Sequence.

Example: If X_n represents the outcome of the nth toss of a fair die, the $\{X_n : n \ge 1\}$ is a discrete random sequence. Since T = {1,2,3, . . . } and S = {1,2,3,4,5,6}

3) **Condition for Stationary Process:** $E\left[X(t)\right] = \text{Constant}, Var\left[X(t)\right] = \text{constant}$.
   If the process is not stationary then it is called evolutionary.

4) **Wide Sense Stationary (or) Weak Sense Stationary (or) Covariance Stationary:**
   A random process is said to be WSS or Covariance Stationary if it satisfies the following conditions.
   i) The mean of the process is constant (i.e) $E\left(X(t)\right) = \text{constant}$.
   ii) Auto correlation function depends only on τ (i.e) $R_{XX}(\tau) = E\left[X(t).X(t+\tau)\right]$

5) **Property of autocorrelation:**

(i) $\left[E\left(X(t)\right)\right]^2 = \lim_{\tau \to \infty} R_{XX}(\tau)$

(ii) $E\left(X^2(t)\right) = R_{XX}(0)$

6) **Markov process:**
A random process in which the future value depends only on the present value and not on the past values, is called a markov process. It is symbolically represented by $P\left[X(t_{n+1}) \le x_{n+1} / X(t_n) = x_n, X(t_{n-1}) = x_{n-1} ... X(t_0) = x_0\right]$

$$= P\left[X(t_{n+1}) \le x_{n+1} / X(t_n) = x_n\right]$$

Where $t_0 \le t_1 \le t_2 \le ... \le t_n \le t_{n+1}$

7) **Markov Chain:**
If for all n, $P\left[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ... X_0 = a_0\right]$
$= P\left[X_n = a_n / X_{n-1} = a_{n-1}\right]$ then the process $\left\{X_n\right\}$, $n = 0,1,2,...$ is called the markov chain. Where $a_0, a_1, a_2, ... a_n, ...$ are called the states of the markov chain.

8) **Transition Probability Matrix (tpm):**
When the Markov Chain is homogenous, the one step transition probability is denoted by P_{ij}. The matrix P = {P_{ij}} is called transition probability matrix.

9) **Chapman – Kolmogorov theorem:**
If 'P' is the tpm of a homogeneous Markov chain, then the n – step tpm $P^{(n)}$ is equal to P^n. (i.e) $P_{ij}^{(n)} = \left[P_{ij}\right]^n$.

10) **Markov Chain property:** If $\Pi = \left(\Pi_1, \Pi_2, \Pi_3\right)$, then $\Pi P = \Pi$ and $\Pi_1 + \Pi_2 + \Pi_3 = 1$.

11) **Poisson process:**
If $X(t)$ represents the number of occurrences of a certain event in $(0,t)$,then the discrete random process $\left\{X(t)\right\}$ is called the Poisson process, provided the following postulates are satisfied.

(i) $P\left[1 \text{ occurrence in } (t, t+\Delta t)\right] = \lambda \Delta t + O\left(\Delta t\right)$
(ii) $P\left[0 \text{ occurrence in } (t, t+\Delta t)\right] = 1 - \lambda \Delta t + O\left(\Delta t\right)$
(iii) $P\left[2 \text{ or more occurrences in } (t, t+\Delta t)\right] = O\left(\Delta t\right)$
(iv) $X(t)$ is independent of the number of occurrences of the event in any interval.

12) **Probability law of Poisson process:** $P\left\{X(t) = n\right\} = \dfrac{e^{-\lambda t}\left(\lambda t\right)^n}{n!}, \quad n = 0,1,2,...\infty$

Mean $E\left[X(t)\right] = \lambda t$, $E\left[X^2(t)\right] = \lambda^2 t^2 + \lambda t$, $Var\left[X(t)\right] = \lambda t$.

Unit – IV (Queueing Models)

n – Number of customers in the system.

λ – Mean arrival rate.

μ – Mean service rate.

P_n – Steady State probability of exactly n customers in the system.

L_q – Average number of customers in the queue.

L_s – Average number of customers in the system.

W_q – Average waiting time per customer in the queue.

W_s – Average waiting time per customer in the system.

Model – I (M / M / 1): (∞ / FIFO)

1) Server Utilization $\rho = \frac{\lambda}{\mu}$

2) $P_n = \rho^n (1-\rho)$ (P_0 no customers in the system)

3) $L_s = \frac{\rho}{1-\rho}$

4) $L_q = \frac{\rho^2}{1-\rho}$

5) $W_s = \frac{1}{\mu(1-\rho)}$

6) $W_q = \frac{\rho}{\mu(1-\rho)}$

7) Probability that the waiting time of a customer in the system exceeds t is $P(w_s > t) = e^{-(\mu-\lambda)t}$.

8) Probability that the quue size exceeds "t" is $P(N > n) = \rho^{n+1}$ where $n = t+1$.

Model – II (M / M / C): (∞ / FIFO)

1) $\rho = \dfrac{\lambda}{\mu s}$

2) $P_0 = \left[\sum_{n=0}^{s-1} \dfrac{(s\rho)^n}{n!} + \dfrac{(s\rho)^s}{s!(1-\rho)} \right]^{-1}$

3) $L_q = \dfrac{1}{s.s!} \dfrac{(s\rho)^{s+1}}{(1-\rho)^2} P_0$

4) $L_s = L_q + s\rho$

5) $W_q = \dfrac{L_q}{\lambda}$

6) $W_s = \dfrac{L_s}{\lambda}$

7) The probability that an arrival has to wait: $P(N \geq s) = \dfrac{(s\rho)^s}{s!(1-\rho)} P_0$

8) The probability that an arrival enters the service without waiting = 1 – P(an arrival hat to wait) = $1 - P(N \geq s)$

9) $P(w > t) = e^{-\mu t} \left\{ 1 + \dfrac{(s\rho)^s \left[1 - e^{-\mu t(s-1-s\rho)} \right]}{s!(1-\rho)(s-1-s\rho)} P_0 \right\}$

Model – III (M / M / 1): (K / FIFO)

1) $\rho = \dfrac{\lambda}{\mu}$

2) $P_0 = \dfrac{1-\rho}{1-\rho^{k+1}}$ (No customer)

3) $\lambda' = \mu(1 - P_0)$ (effective arrival rate)

4) $L_s = \dfrac{\rho}{1-\rho} - \dfrac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$

5) $L_q = L_s - \dfrac{\lambda'}{\mu}$

6) $W_s = \frac{L_s}{\lambda'}$

7) $W_q = \frac{L_q}{\lambda'}$

8) $P[\text{a customer turned away}] = P_k = \rho^k P_0$

Model – IV (M / M / C): (K / FIFO)

1) $\rho = \frac{\lambda}{\mu s}$

2) $P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!} \sum_{n=s}^{k} \rho^{n-s} \right]^{-1}$

3) $P_n = \begin{cases} \frac{(s\rho)^n}{n!} P_0, & n \le s \\ \frac{(s\rho)^n}{s!\, s^{n-s}} P_0, & s \le n \le k \end{cases}$

4) Effective arrival rate: $\lambda' = \mu \left[s - \sum_{n=0}^{s-1} (s-n) P_n \right]$

5) $L_q = \frac{(s\rho)^s}{s!} \left[\frac{\rho\left(1-\rho^{k-s}\right)}{(1-\rho)^2} - \frac{(k-s)\rho^{k-s+1}}{1-\rho} \right] P_0$

6) $L_s = L_q + \frac{\lambda'}{\mu}$

7) $W_q = \frac{L_q}{\lambda'}$

8) $W_s = \frac{L_s}{\lambda'}$

Unit – V (Advanced Queueing Models)

1) **Pollaczek – Khintchine formula:**

$$L_S = \lambda E(t) + \frac{\lambda^2 \left[Var(t) + \left(E(t) \right)^2 \right]}{2\left[1 - \lambda E(t)\right]}$$

(or)

$$L_S = \rho + \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}$$

2) **Little's formulas:**

$$L_S = \rho + \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}$$

$$L_q = L_S - \rho$$

$$W_S = \frac{L_S}{\lambda}$$

$$W_q = \frac{L_q}{\lambda}$$

3) **Series queue (or) Tandem queue:**

The balance equation

$$\lambda P_{00} = \mu_2 P_{01}$$

$$\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$$

$$\lambda P_{01} + \mu_2 P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$$

$$\mu_1 P_{11} + \mu_2 P_{11} = \lambda P_{01}$$

$$\mu_2 P_{b1} = \mu_1 P_{11}$$

Condition $P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$

4) **Open Jackson networks:**

i) Jackson's flow balance equation $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$

Where k – number of nodes, r_j – customers from outside

ii) Joint steady state probabilities

$$P(n_1, n_2, \ldots n_k) = \rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)\ldots\rho_k^{n_k}(1-\rho_k)$$

iii) Average number of customers in the system

$$L_S = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \ldots + \frac{\rho_k}{1-\rho_k}$$

iv) Average waiting time of a customers in the system

$$W_S = \frac{L_S}{\lambda} \qquad \text{where } \lambda = r_1 + r_2 + \ldots + r_k$$

5) **Closed Jackson networks:**

In the closed network, there are no customers from outside, therefore $r_j = 0$ then

i) The Jackson's flow balance equation $\lambda_j = \sum_{i=1}^{k} \lambda_i P_{ij}$ $\quad \because r_j = 0$

(or)

$$(\lambda_1 \ \lambda_2 \ \ldots \ \lambda_k) = (\lambda_1 \ \lambda_2 \ \ldots \ \lambda_k)\begin{pmatrix} P_{11} & P_{12}\ldots & P_{1k} \\ P_{21} & P_{22}\ldots & P_{2k} \\ \vdots & \vdots & \vdots \\ P_{k1} & P_{k2}\ldots & P_{kk} \end{pmatrix}$$

ii) If each nodes single server

$$P(n_1, n_2, \ldots n_k) = C_N \rho_1^{n_1} \rho_2^{n_2} \ldots \rho_k^{n_k}$$

Where $C_N^{-1} = \sum_{n_1+n_2+\ldots+n_k=N} \rho_1^{n_1} \rho_2^{n_2} \ldots \rho_k^{n_k}$

iii) If each nodes has multiple servers

$$P(n_1, n_2, \ldots n_k) = C_N \frac{\rho_1^{n_1}}{a_1} \frac{\rho_2^{n_2}}{a_2} \ldots \frac{\rho_k^{n_k}}{a_k}$$

Where $C_N^{-1} = \sum_{n_1+n_2+\ldots+n_k=N} \frac{\rho_1^{n_1}}{a_1} \frac{\rho_2^{n_2}}{a_2} \ldots \frac{\rho_k^{n_k}}{a_k}$

$$a_i = \begin{cases} n_i! & , n_i < s_i \\ s_i!\, s_i^{n_i - s_i} & , n_i \ge s_i \end{cases}$$

---- *All the Best* ----